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Estimation of geometric Brownian motion model with a *t*-distribution–based particle filter

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Orientation: Geometric Brownian motion (GBM) model basically suggests whether the distribution of asset returns is normal or lognormal. However, many empirical studies have revealed that return distributions are usually not normal. These studies, time and again, discover evidence of non-normality, such as heavy tails and excess kurtosis.

Research purpose: This work was aimed at analysing the GBM with a sequential Monte Carlo (SMC) technique based on *t*-distribution and compares the distribution with normal distribution.

Motivation for the study: The SMC or particle filter based on the *t*-distribution for the GBM model, which involves randomness, volatility and drift, can precisely capture the aforementioned statistical characteristics of return distributions and can predict the random changes or fluctuation in stock prices.

Research approach/design and method: The particle filter based on the *t*-distribution is developed to estimate the random effects and parameters for the extended model; the mean absolute percentage error (MAPE) were calculated to compare distribution fit. Distribution performance was assessed through simulation study and real data.

Main findings: Results show that the GBM model based on student's *t*-distribution is empirically more successful than the normal distribution.

Practical/managerial implications: The proposed model which is heavier tailed than the normal does not only provide an approximate solution to non-normal estimation problem.

Contribution/value-add: The GBM model based on student's *t*-distribution establishes an efficient structure for GBM and volatility modelling.

Introduction

Most of the models utilised in the description of financial time series are written in terms of a continuous time diffusion S_t that satisfies the stochastic differential equation (SDE):

$$dS_t = \mu S_t d_t + \sigma S_t \, dB_t \qquad \text{[Eqn 1]}$$

where $dB_t \sim N(0,dt)$ is the increment to Brownian motion process, and σS_t and μS_t denote the volatility and drift function, respectively. This class of parametric models has been extensively used to portray the dynamics of financial variables, including stock prices, interest rates and exchange rates. A stochastic process S_t is said to follow a geometric Brownian motion (GBM) if it satisfies the above SDE.

The GBM is one of the most popular stochastic processes and undoubtedly an effective instrument in modelling and predicting the random changes in stock prices that evolve over time. It is essentially useful for index price study because the process in question assumes that percentage changes are independent and identically distributed over equal and non-overlapping time length (Luenberger 1995; Ross 2000). The GBM model assumes that the instantaneously expected rate of return is constant. Hence, the constant instantaneous expected drift assumption of the standard Brownian process is substituted with the constant expected rate of return in the geometric Brownian process (Hull 2000). The GBM model usually assumes that the distribution of asset returns is either normal or lognormal. However, financial data often have heavier tails than can be captured by the standard GBM model. As such, there is a need to use non-normal distributions to better model and deal with the heavy tails (Carol 2004; Tan 2005; Tan & Chu 2012; Tan & Tokinaga 2006, 2007). However, intractable likelihood functions for SDEs make inference challenging, necessitating the requirement of simulation-based techniques to estimate and maximise the likelihood function. The sequential Monte Carlo (SMC) methods or the particle filter have allowed for the accurate evaluation of likelihoods at fixed parameter values (Nemeth, Feamhead & Mihaylova 2014).

In this article, we extended our investigations by introducing a GBM model based on the *t*-distribution–based particle filter to approximate the return distributions of assets and compared the distribution with normal distribution. In evaluating the proposed GBM level of precision, the model parameters are estimated. An SMC or particle filter technique based on student's *t*-distribution is developed to estimate the parameters for the extended model. The ensuing models are applied to modelling the closing stock price of five firms of the Nigerian Stock Exchange (NSE).

Review of relevant literature

A number of studies have been conducted in the area of GBM as a model for stock prices. Some scholars have tried extending, and hence improving, the standard GBM model. Duplantier (2005) referred to Louis Bachelier, who in his PhD thesis in 1900 mentioned that the stock price dynamics follow a Brownian motion. The process he applied can produce shares that could allow both negative security prices and option prices that exceed the price of the underlying asset. Osborne (1959) refined the Bachelier model by employing the stochastic exponential of the Brownian motion to model stock price. Samuelson (1965) extended the GBM by using the discount rate in pricing. For him, the return rates, instead of the stock prices, follow the GBM (Piasecki 2006). Some scholars represented rare events by jumps and introduced a model of jump diffusion (see Kou 2002; Merton 1976). Others presented a more realistic stochastic process for the underlying process (e.g. stock price) by bringing in a stochastic process for the volatility, that is, with the variance of the stock return as random (e.g. Heston 1993; Hull & White 1987; Stein & Stein 1991).

Thao (2006) tried to replace the Brownian motions with fractional Brownian motions in the diffusion model. Sattayatham, Intrasit and Chaiyasena (2007) improved on Thao's results by adding a Poisson jump into the model.

Geometric Brownian motion has been extensively used as a model for stock prices, commodity prices, growth in demand for products and services, and real options analysis (Benninga & Tolkowsky 2002; Nembhard, Shi & Aktan 2002; Thorsen 1998). It has also been used for representing future demand in capacity studies (Lieberman 1989; Ryan 2006; Whitt 1981). On the whole, its acceptance was motivated from the assumption that random changes over time follow a GBM process (Marathe & Ryan 2005). On the other hand, some scholars have raised relevant questions concerning the accuracy of the GBM (e.g. Marathe & Ryan 2005; Ross 1999; Thorsen 1998; Watteel-Sprague 2000).

Works on modelling return distributions of financial assets also exist. The most used are the normal, the lognormal and the non-Gaussian stable distributions. Other types of distributions, such as the student's *t*, the skewed student's *t*, the generalised *t*, the generalised error distribution (GED), the skewed GED and mixture distribution of Gaussian distributions, have also been applied. The normal distribution is one of the most usually applied distributions. It was extensively used in the 1700s; in 1800, Karl Gauss successfully applied it to astronomical data analysis. It became known as the Gaussian distribution. Empirical analyses from the late 1960s were not successful in supporting the normal assumption on estimating the return distribution of real financial data. Mandelbrot (1963) affirmed that although financial prices or its logarithm following a Brownian motion is mathematically convenient, it is difficult to fit the real financial data with this assumption. Fama (1965) analysed equilibrium asset pricing and noted that the daily return distribution follows a non-Gaussian distribution. Both Mandelbrot and Fama pointed out that excess kurtosis and heavy tails exist in real financial data.

Hsu, Miller and Wichern (1974) and Hagerman (1978) showed from their studies that return distributions are non-normal. Bollerslev (1987) found leptokurtosis in monthly Standard & Poor's 500 Index returns. Kariya et al. (1995) and Nagahara (1996) revealed that the return distributions of Japanese stocks are fat-tailed and skewed. Kitagawa, Sato and Nagahara (1999) found that daily or weekly return distributions are not normal but fat-tailed and skewed according to observed financial data. Harvey and Siddique (2000) and Premaratne and Bera (2000) confirmed that the asymmetry of return distribution exists in real business data. Gerig, Vicente and Fuentes (2009) presented a model that explained the shape and scaling of the distribution of intraday stock price fluctuations and verified the model by using a large database made up of several stocks traded on the London Stock Exchange. Their findings showed that the return distributions for these stocks are non-Gaussian, similar in shape and appear to be stable over intraday time scales.

Theodossiou (1998) advocated the use of a skewed generalised *t*-distribution, which combines the student's *t* and skewed student's *t*, to model return distributions. Furthermore, Theodossiou (2000) pointed out that a skewed GED fits the financial data well, while asymmetry and excess kurtosis are observed in the financial data.

Methodology

Geometric Brownian motion model

Geometric Brownian motion is the stochastic process used in the Black–Scholes methodology to model the evolution of prices in time. As in a typical structural model, let us consider a firm with its value of the asset V_t following a GBM:

$$dV_t = \mu V_t d_t + \sigma_v V_t dB_t \qquad \text{[Eqn 2]}$$

where μ and σ are drift and volatility parameters to be estimated. The drift informs us on the average rate at which a value increases in a stochastic process, while the volatility is the constant characteristic of the stock prices that tells us the measure of the fluctuations of the stock prices. Relatively high volatility means that the stock price varies continuously within relatively large intervals. The notation dt is an infinitely approaching 0 time difference between time points t and t - 1 and the last term involves random $dB_t \sim N(0, dt)$ increment to Brownian motion process. The right-hand side term $\mu V_t d_t$ controls the 'trend' of this trajectory and the term $\sigma V_t dB_t$ controls the random noise in the trajectory. Nevertheless, one of the foremost challenges in applying this model to financial market data is the fact that the underlying asset value process is unobservable.

Applying the Ito's formula (see Lamberton & Lapeyre 1997) on equation (2) with $F(S) = \ln S$, we obtain the following equation:

$$\ln S_t = \left[\mu - \frac{1}{2}\sigma^2\right]t + \sigma B_t \quad \text{[Eqn 3]}$$

The stochastic process, as characterised by equation (3), indicates that ln*S* is normally distributed. Equivalently, *S* is lognormally distributed.

Taking the exponential of both sides and inserting the initial condition S_0, we obtain the solution. The analytical solution of this GBM is given by:

$$S_t = S_0 \exp((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t \quad \text{[Eqn 4]}$$

This SDE is principally significant in the modelling of many asset classes. Equation (4) is the asset price model that can predict an asset price at specific time *t*. We can represent GBM solution as follows:

$$S_t = S_0 e^{X_t} \quad \text{[Eqn 5]}$$

where $X_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$.

Geometric Brownian motion model maximum likelihood estimation

The parameters μ and σ can be estimated using historical data for stock price, bearing in mind that the time difference for data with monthly frequency is $\Delta t = \frac{1}{12}$.

As Brigo et al. (2007) noted, the parameters that need to be optimised are $\theta = (\mu, \sigma)$ for the GBM.

The likelihood function is denoted as:

$$L(\theta) = f_\theta(y_{t_1}, y_{t_2}, \ldots, y_{t_n})$$

$$= \prod_{i=1}^{n} f_\theta(y_{t_i})$$

$$= \prod_{i=1}^{n} f(y_{t_i} | \theta)$$

where f_θ is the probability density function, and $y_{t_1}, y_{t_2}, y_{t_n}$ are the log returns.

Let $\theta = (\mu, \sigma)$, then the probability density function f_θ is:

$$f_\theta(y_{t_i}) = \frac{1}{x_{t_i}\sigma\sqrt{2\pi t}} \exp - \frac{\left(\left(\frac{y_{t_i}}{y_{t_o}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}$$

The likelihood function is maximised to get the optimal estimators $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$.

As stated earlier, the Wiener process dB_t is assumed to follow a normal distribution with mean 0 and variance *dt*. From the corresponding continuous density function of the Wiener process, the natural logarithm of the likelihood function is obtained:

$$f(dB_t) = \frac{1}{\sqrt{2\pi dt}} e^{-\frac{1}{2dt}(dB_t)^2}$$

$$f(dB_1,\ dB_2, \ldots dB_1) = f(dB_1)\, f(dB_2)\, f(dB_3) \ldots f(dB_t)$$

$$\prod_{t=1}^{T} \frac{1}{\sqrt{2\pi dt}} \exp\left[-\frac{1}{2dt}(dB_t)^2\right]$$

$$\ln L(\theta) = \sum_{t=1}^{T}\left[-\frac{1}{2}\ln(2\pi\, dt) - \frac{(dB_t)^2}{2dt}\right]$$

$$\ln L(\theta) = -\frac{T}{2}\ln(2\pi\, dt) - \frac{1}{2\,dt}\sum_{t=1}^{T}(dB_t)^2$$

$dB_t \sim N(0, dt) \Leftrightarrow Var(dB) = dt$

let $dt = 1$

The expression $\left[-\frac{T}{2}\ln(2\pi\, dt)\right]$ is unrelated to the parameters under study within the $L(\theta)$. Thus, the $\left[-\frac{T}{2}\ln(2\pi\, dt)\right]$ expression does not hold any importance for the parameters estimation and is removed from the log-likelihood function; thus, its transformed version gives:

$$\ln L_{Tr}(\theta) = \frac{1}{2dt}\sum_{t=1}^{T} (dB_t)^2$$

From the above, the specific log-likelihood function of the GBM is found by replacing the equivalent Weiner process such that:

$$\ln L_{Tr}(\mu,\sigma) = \frac{1}{2}\sum_{t=1}^{T}\left[\frac{\ln\left(\frac{y_{t_1}}{y_{t_0}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma}\right]$$

The natural logarithm of the likelihood function is differentiated in terms of μ and σ and then equated to zero to give equations:

$$\hat{y} = \left(\hat{\mu} - \frac{1}{2}\hat{\sigma}^2\right)\Delta t \quad \text{[Eqn 6]}$$

$$\hat{s} = \hat{\sigma}^2 \Delta t \quad \text{[Eqn 7]}$$

where

$$\hat{y} = \frac{\sum_{i=1}^{n} y_{t_i}}{n}$$

$$\hat{v} = \frac{\sum_{i=1}^{n}(y_{t_i} - \hat{y})^2}{n}$$

Determining $\hat{y}$ and $\hat{s}$, the corresponding maximum likelihood estimator (MLE) of μ and σ are $\hat{\sigma}^2 = \frac{\hat{s}}{\Delta t}$ and $\hat{\mu} = \frac{1}{2}\hat{\sigma}^2 + \frac{\hat{y}}{\Delta t}$

Geometric Brownian motion with t-distribution

Asset return distributions are frequently presumed to follow either a normal or a lognormal distribution. It can also follow GBM based on the Gaussian process. However, many empirical studies have shown that return distributions are usually not normal. They often find evidence of non-normality, such as heavy tails, excess kurtosis and finite moments. One class of fat-tailed distributions with the potential to give a better approximation to the distribution of stock returns is the *t*-distribution.

An extension of the version of the GBM model, wherein it is assumed that the random noise process, dB_t, is a student's *t*-distribution, is considered. The proposed student's *t*-distribution with degrees of freedom, *v*, for the last term, dB_t, effects a change in the equation:

$$dV_t = \mu V_t d_t + \sigma V_t dB_t \quad dB_t \sim t_v,\ t = 1,\ldots,n. \qquad \text{[Eqn 8]}$$

The distribution of the error term for this specification according to Shimada and Tsukuda (2005) assumes the following form:

$$f_\theta(y_t) = \frac{1}{\sqrt{\pi(v-2)}} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} e^{-\frac{y_t}{2}} \left(1 + \frac{y_t^2 e^{-x_t}}{v-2}\right)^{-\frac{v+1}{2}}$$

Sequential Monte Carlo algorithm analysis

The SMC, also known as particle filter algorithm (Gordon, Salmond & Smith 1993), in sequential estimation on hidden asset value and model parameters estimation, is applied under the GBM model. This method applies the concept of sampling-importance-resampling (SIR) (Rubin 1987). One of the key challenges in applying structural models to financial market data is the fact that the underlying asset value process is unobservable. Furthermore, at each time *t*, market values of stock are known only up to the time *t*, which means that the information needs to be updated sequentially. In this section, with known model parameters, we apply the particle filter algorithm to update the information about the underlying asset value process recursively from the observed times series. By running the filtering algorithm, the conditional distribution of the underlying asset value is approximated and recursively updated, given observed prices.

Assuming that we have at time *t* weighted particles $\{f_t^{(i)}, w_t^{(i)}\}$ drawn from $f(x_t \mid y_t)$, $f_t^{(i)}$ is a set of particle filters with associated weight $w_t^{(i)}$. This is seen as an empirical approximation for the density made up of point masses:

$$f(x_t \mid y_t) \approx \sum_{i=1}^{M} w_t^{(i)} \delta(x_t - f_t^{(i)}). \qquad \text{[Eqn 9]}$$

Kitagawa and Sato (2001) and Kitagawa (1996) offer an algorithm for filtering as follows:

(1) For $i = 1, \ldots, N$, generate a random number $f_0^{(i)} \sim p(x_0)$

(2) Repeat the following steps for $t = 1, \ldots, T$.

(a) For $i = 1, \ldots, N$, generate a random number $w_t^{(i)} \sim q(w)$

(b) For $i = 1, \ldots, N$, compute $p_t^{(i)} = F(f_{t-1}^{(i)}, w_t^{(i)})$

(c) For $i = 1, \ldots, N$, compute $w_t^{(i)} = p(y_t \mid p_t^{(i)})$

(d) Generate $f_t^{(i)}, i = 1, \ldots N$ by resampling $p_t^{(i)}, \ldots, p_t^{(N)}$

(3) This Monte Carlo filter returns

$$\{f_t^{(i)}, i = 1,\ldots,N, t = 1,\ldots,m\} \text{so that} \sum_{i=1}^{N} \frac{1}{N} \delta(x_t - f_t^{(i)}) \approx f(x_t \mid Y_t).$$

Estimation procedure

In this section, with known model parameters, we apply the particle filter algorithm based on t-distribution to update the information about the underlying asset value process recursively from the observed times series of stock prices.

With known parameters $\Theta = \{\mu, \sigma\}$, we observe the time series of stock prices $S = \{S_t;\ t = 1, \ldots T\}$ and have the hidden asset process to be estimated $V = \{V_t;\ t = 1, \ldots T\}$. The algorithm is as follows:

The algorithm for the filtering is an extension of Godsill, Doucet and West's (2004) and Kim and Stoffer's (2008) algorithms. From here *M* samples from $f(V_t \mid S_t)$ for each *t* were obtained.

(1) Given $\{V_t^{(i)}\}_{i=1}^{m}$, draw $\{V_{t+1}^{(*i)}\}$ from $p_{t+1}(V_{t+1} \mid V_t^{(i)}) \sim t_v$ for $i = 1,\ldots, M$

Generate $f_0^{(i)} \sim t_v$

(2) Generate a random number $w_t^{(i)} \sim t_v$,

(3) Compute $p_t^{(i)} = \mu f_{t-1}^{(i)} + \sigma w_t^{(i)}$

(a) Compute $w_t^{(i)} = p(s_t \mid p_t^{(i)},) \propto e^{-\frac{s_t}{2}} \left(1 + \frac{s_t^2 e^{-s_t}}{v-2}\right)^{-\frac{v+1}{2}}$

(b) Generate $f_t^{(i)}$ by resampling with weights, $w_t^{(j)}$

Resample from $\{(V_{t+1}^{(*1)}), (V_{t+1}^{(*2)}), \ldots, (V_{t+1}^{(*M)})$ with probability proportional to $w_t^{(j)}$.

Evaluation methods of geometric Brownian motion

As averred by Lawrence, Klimberg and Lawrence (2009), we have three measurements of forecasting model which involve time period *t*. The measurements are number of period forecasts, *n*; actual value in time period at time *t*, Y_t; and forecast value at time period *t*, F_t. The mean absolute percentage error (MAPE) seems to be the most widely used to evaluate the forecasting method that considers the effect of the magnitude of the actual values. It is a measure of prediction accuracy of a forecasting method in statistics. It usually expresses accuracy as a percentage and is defined as follows:

$$MAPE = 100 \times \frac{1}{n} \sum \left| \frac{Y_t - F_t}{Y_t} \right|$$

The difference between Y_t and F_t is divided by the actual value Y_t again. The absolute value in this calculation is summed for every forecasted point in time and divided by the number of fitted points n. Multiplying by 100 makes it a percentage error.

Empirical results and discussion

Data

We apply the above-described methodology to model the stock prices of five firms of the NSE – each from five different sectors, namely the banking sector (Guaranteed Trust Bank [GTB]), oil and gas sector (Oando), construction sector (Julius Berger [Jberger]), health care sector (GlaxoSmith) and industrial goods sector (Chemical & Allied Product [CAP]) over the period 02 January 2010 to 31 December 2014.

The data series is transformed into daily log returns series so that we can obtain stationary series. Descriptive statistical summary is obtained to view the data for the daily stock prices and returns of all the indices.

Table 1 reports the descriptive statistics for all the five selected indices. The value of the kurtosis for returns is high and greater than 3. This shows that the distribution is leptokurtic, that is, it is fat-tailed and that the returns display financial characteristics of volatility clustering and leptokurtosis. The skewness for both prices and returns is positive, showing that the distribution has a long right tail. The high values of kurtosis for the returns indicate that extreme price changes occurred frequently during the sampling period. The positive skewness and kurtosis indicated non-normal series. With reference to Jarque-Bera statistics, the stock index series is non-normal at 99% confidence interval as the probability is 0×10^{-6}, which is less than 0.01.

Figures 1–5 show the plot of each of the five firms' stock prices.

The stock prices of each of the five firms of the NSE for the year 2010–2014 were used to derive the drift and volatility. Table 2 shows the observed values.

These two parameters (drift and volatility) were then used to create the geometric Brownian path for both the GBM normal and student's t-distribution of each of the five firms of the NSE. We compute the MLEs and the corresponding log-likelihood for each stock. Table 3 summarises the estimated parameters for both the GBM normal and student's t-distribution.

The log-likelihood for the GBM student's t-distribution model is higher than that of the model for the GBM normal for each of the five stock series. Comparing the two models based on the Akaike information criterion (AIC), it is found that the GBM t-distribution model outperforms the GBM normal model for each of the five stocks.

The GBM normal and student's t–based particle filter method are then run on the simulated prices process, and the average MAPE is calculated. Both models estimate the volatility

TABLE 1: Descriptive statistical summary for the daily stock prices.

Index	Prices					Returns				
	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera	Mean	Standard deviation	Skewness	Kurtosis	Jarque-Bera
Guaranteed Trust Bank (GTB)	40.3754	18.4573	0.6718	3.3487	194.387 (0.0000)	0.0030950	0.0278	-2.4739	29.4686	684.195 (0.0000)
Oando	70.9741	47.4833	1.0297	4.8445	202.394 (0.0000)	-0.00062810	0.0352	-1.3034	54.4961	312.573 (0.0000)
Julius Berger (JBerger)	46.8880	25.5413	0.9931	4.2569	207.281 (0.0000)	0.0041886	0.0458	-1.5541	42.6513	497.263 (0.0000)
GlaxoSmith	76.9873	48.2532	1.1542	4.2342	213.237 (0.0000)	0.0070399	0.0300	-0.1911	30.2326	513.240 (0.0000)
Julius Berger (CAP)	49.3441	27.3775	0.2783	3.6845	195.142 (0.0000)	0.000875734	0.0367	-3.6781	20.7944	795.142 (0.0000)

GTB, Guaranteed Trust Bank; JBerger, Julius Berger; CAP, Chemical & Allied Product.

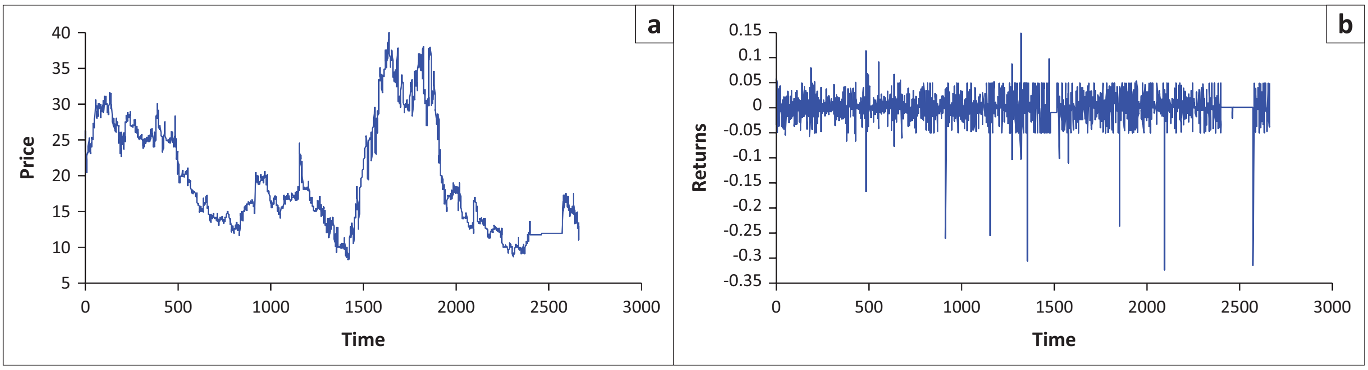

FIGURE 1: Guaranteed Trust Bank (GTB) (a) stock pricing over time, (b) returns over time.

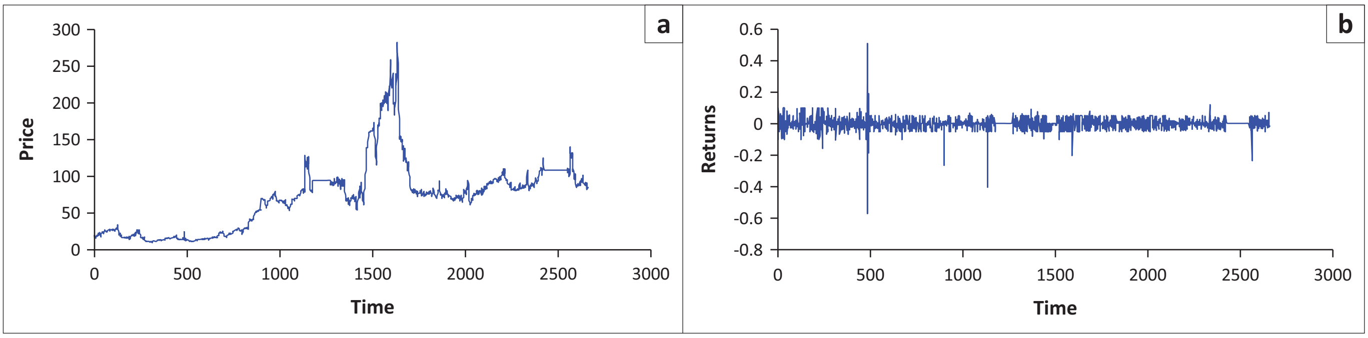

FIGURE 2: Oando (a) stock pricing over time, (b) returns over time.

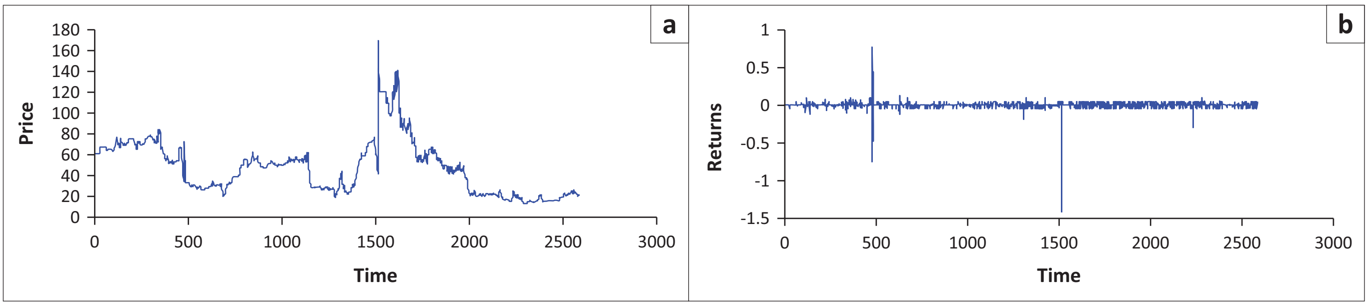

FIGURE 3: JBerger (a) stock pricing over time, (b) returns over time.

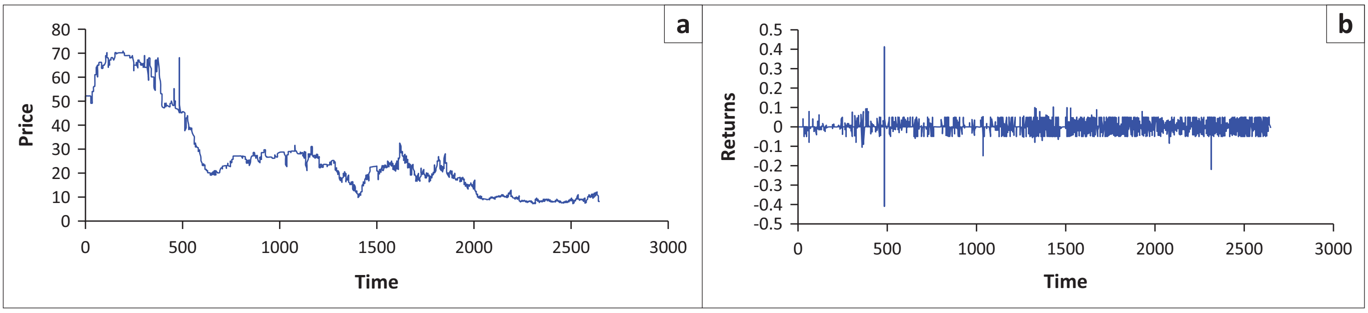

FIGURE 4: GlaxoSmith (a) stock pricing over time, (b) returns over time.

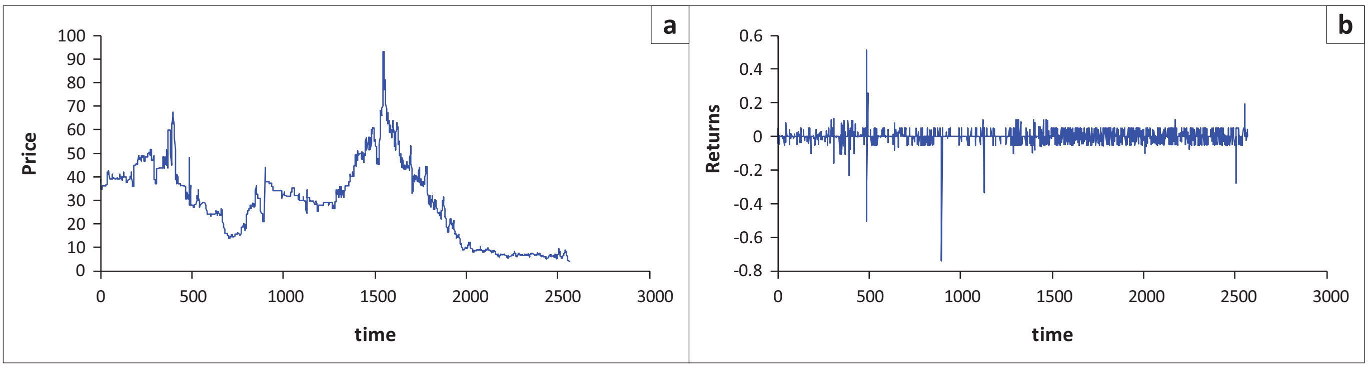

FIGURE 5: Chemical and Allied Product (CAP) (a) stock pricing over time, (b) returns over time.

TABLE 2: Drift and volatility values of stock prices.

Index	Drift (μ)	Volatility (σ)
GTB	0.0720	0.2816
Oando	0.0485	0.2794
JBerger	0.0514	0.2723
GlaxoSmith	0.0354	0.2837
CAP	0.0624	0.2808

GTB, Guaranteed Trust Bank; JBerger, Julius Berger; CAP, Chemical & Allied Product.

process of each of the five firms' stock prices using known parameters. Table 4 presents the observed values.

Graphically, for a single run, the estimation results obtained from running these two models are shown in Figures 6–10. Each figure shows a GBM's normal paths (blue line), student's t paths (red line) and the actual volatility price curve (black line) of each of the five firms' stock prices.

TABLE 3: Estimated parameters of the geometric Brownian motion normal and student's *t*-distribution.

Index	GBM normal				GBM student's *t*			
	μ	σ	Log- likelihood	AIC	μ	σ	Log-likelihood	AIC
GTB	0.45	0.37	-2797	4359	0.34	1.09	-2740	4248
Oando	0.12	0.33	-2341	4686	0.30	1.27	-2331	4568
JBerger	0.10	0.40	-2149	4302	0.37	1.03	-2135	4176
GlaxoSmith	0.23	0.36	-2344	4255	0.41	1.04	-1234	4234
CAP	0.34	0.44	-2783	4684	0.32	1.23	-1345	4221

CAP, Chemical & Allied Product, JBerger, Julius Berger; AIC, Akaike information criterion; GBM, geometric Brownian motion; GTB, Guaranteed Trust Bank.

TABLE 4: Evaluation statistic-distribution comparison of techniques based on the normal and student's *t*.

Models	Mean absolute percentage deviation
GBM normal	0.0967 = 9.67%
GBM student's *t*	0.0652 = 6.52%

GBM, Geometric Brownian motion.

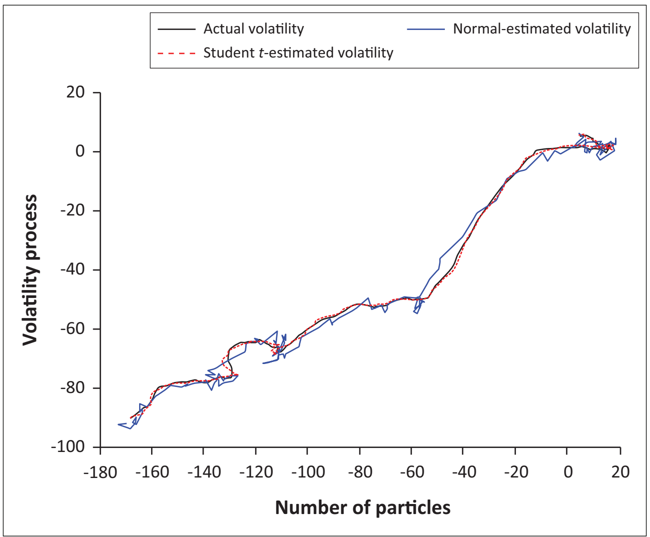

FIGURE 6: Brownian path for Guaranteed Trust Bank (GTB).

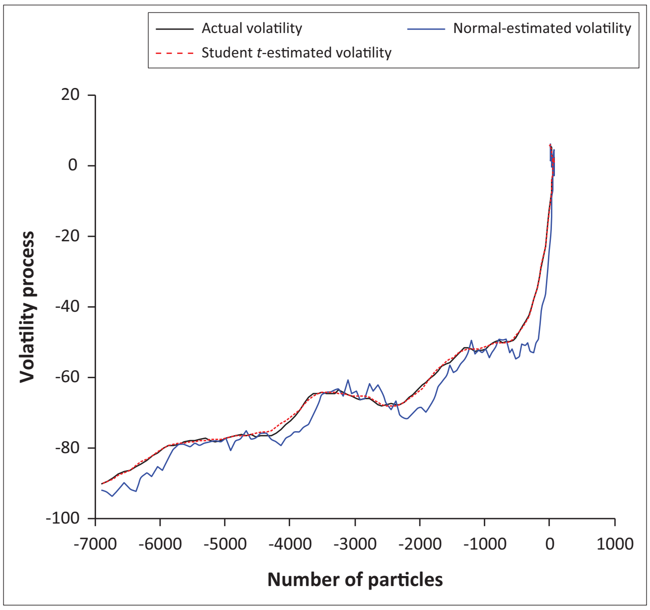

FIGURE 7: Brownian path for Oando.

Figures 6–10 show the plots of volatility estimation for each of the five firms' stock prices. The plots show that the GBM student's *t*–based particle filter estimate (red line) and the actual volatility (black line) lie close to each other compared to the GBM normal estimate (blue line).

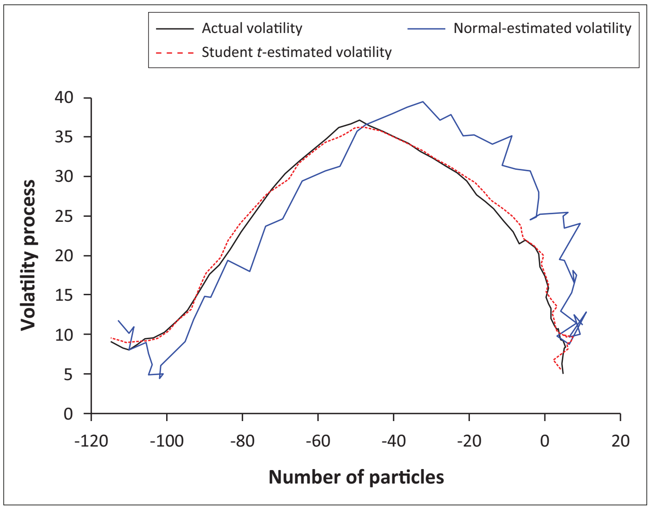

FIGURE 8: Brownian path for JBerger.

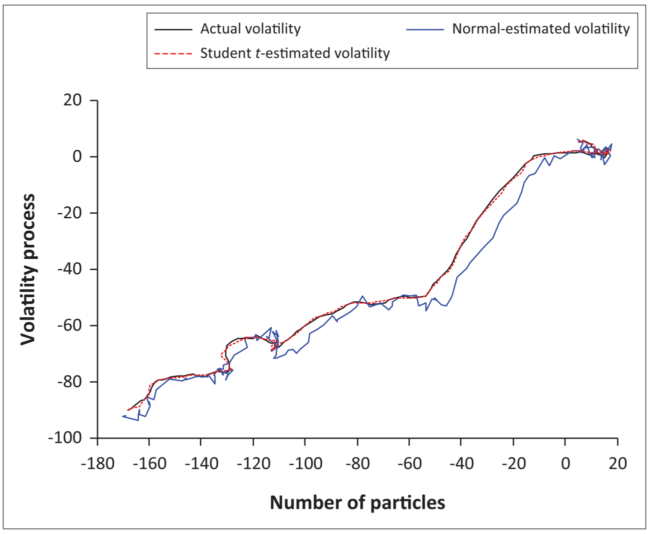

FIGURE 9: Brownian path for GlaxoSmith.

Discussion

Geometric Brownian motion model usually assumes that the distribution of asset returns is either normal or lognormal. Previous approaches to the estimation of GBM model have revealed that return distributions are usually not normal. In this article, a GBM model based on the *t*-distribution–based particle filter to approximate the return distributions of assets is introduced, and a comparison of the distribution and normal distribution is done.

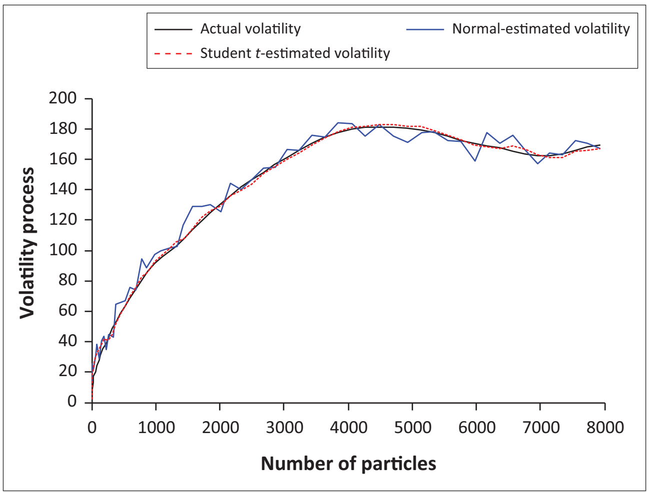

FIGURE 10: Brownian path for Chemical & Allied Product.

The descriptive statistics for the five indices of the NSE are given in Table 1. The highest average returns are obtained in GlaxoSmith. The other indices, which show higher mean returns, are JBerger and GTB. This reflects the performance of these indices. Furthermore, JBerger has the highest standard deviation (0.045), which represents higher volatility, while the lowest standard deviation belongs to GTB (0.027). The returns of the five selected indices are negatively skewed, indicating that the returns are flatter to the left compared to the normal distribution. The significant kurtosis indicates that return distribution has sharp peaks compared to a normal distribution. The Jarque-Bera (1980) statistics confirmed that index returns are non-normally distributed.

In evaluating the proposed GBM level of precision, the model parameters are estimated. A particle filter technique based on student's *t*-distribution is developed to estimate the parameters for the GBM model. Our goal was to considerably improve the forecasting performance of a GBM model using student's *t* particle filter, which is heavier tailed than the normal distribution. The stock prices of each of the five firms were used to derive the drift and volatility parameters (see Table 2). These two parameters were then used to create the geometric Brownian path for both the GBM normal and student's *t*-distribution of each of the five firms of the NSE. Table 3 presents the estimation results along with their log-likelihood and the AIC for the student's *t* and the normal GBM models. The AIC and the log-likelihood values highlight the fact that the GBM student's *t*-distribution better estimates the series than the GBM normal distribution. In fact, the log-likelihood of -2740, -2331, -2135, -1234 and -1345 for the GBM student's *t*-distribution model increases, leading to AIC of 4248, 4568, 4176, 4234 and 4221 versus 4359, 4686, 4302, 4255 and 4684 for the normal GBM model. The evaluation statistics in terms of MAPE indicate that the GBM student's *t*-distribution is highly accurate and the predictive exactness is higher, as proved by the MAPE value, which is lower than 10% (see Table 4).

The plot of volatility estimation for each of the five firms' stock prices is shown in Figure 6–10. The graphical presentation shows three scenarios for the GBM model – the representation of the trajectories of the actual volatility path (shown by black line), the GBM normal (shown by blue line) and the GBM student's *t* estimate (shown by red dotted line) – and infers that the estimation of the GBM student's *t* significantly outperforms the normal GBM model by accurately tracking the actual volatility path for each of the stock prices.

Conclusion

This work presented an extension of the random noise process, dB_t, in the GBM model from normal to student's *t*-distribution. The goal was to compare and contrast the two models in five different stock market periods in terms of their predictability of such exceptional movements in the NSE market. The study revealed that the student's *t* GBM performed better than the normal GBM in estimating both the volatility and the parameters of the model. A particle filter technique based on student's *t*-distribution is developed to estimate the random effects and parameters for the extended model. The functions provided by MATLAB enabled us to develop the techniques based on the student's *t* GBM model and a strategy for fitting the model. This change to the proposed model allows for a more robust fit, giving us a new tool to explore the tail fit. The student's *t* GBM model was compared and evaluated with the normal GBM model. The experimental outcome of the simulation and real data analyses confirms the viability of the proposed method. The evaluation statistics are calculated to compare the fit of distributions. Student's *t* GBM is more highly accurate than the normal GBM as proved by the MAPE value, which is lower than 10%. The results, based on daily stock prices, reveal that the student's *t* GBM is comparable to the normal GBM model but empirically is more successful.

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Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this manuscript.

Authors' contributions

B.E.N. was the project leader and was responsible for most of the theoretical and experimental work done. B.E.N. contributed by introducing an extension of the random noise process in the GBM model from normal to student's *t*-distribution. B.E.N. also introduced techniques based on the student's *t*-distribution and a strategy for the estimation of parameters of the GBM model by using the particle filtering algorithm, there by expanding the scope of application of the GBM model. O.A. made conceptual contributions and wrote and ran the codes for the simulation study.

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